ON THE THEORY OF ROLLING WAVES

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The problem of rolling waves in a sheet of fluid flowing in a vertical plane [1] is treated on the basis of the complete Navier–Stokes equations with conditions on the unknown free boundary. The existence of a one-parameter family of rolling waves, bifurcating from the Poiseuille flow, is proved.

1. STATEMENT OF THE PROBLEM

It is well known that one of the possible regimes of flow of a fluid sheet in a vertical plane under the influence of gravity is plane motion with rectilinear trajectories and a plane free surface (this is hereafter called the fundamental motion). We choose as the units of length, time, velocity, and pressure the quantities b, b/V, V, and ρV^2, respectively. Here b is the thickness of the sheet, $V = gb^2/3\nu$ is the average value of the longitudinal velocity over the thickness of the sheet, ρ is the density, ν is the viscosity of the fluid, and g is the acceleration due to gravity. In dimensionless variables the velocity $\vec{V}$ and the pressure P of the fundamental motion are of the form

$$\vec{V} = [3(1 - x_2^2)/2, 0] \quad P = 0.$$

The line $x_2 = 1$ corresponds to the "bottom," and the line $x_2 = 0$ corresponds to the free boundary.

We shall look for plane motions of the traveling-wave type, which bifurcate from the fundamental motion. We look for velocity and pressure fields in the form $\tilde{\vec{v}} = \vec{V} + \vec{v}$, $\tilde{p} = p/\mathrm{Re}$, where

$$\vec{v} = (u(x, y),\ v(x, y));\ p = p(x, y);\ x = x_1 - ct;\ y = x_2;$$

c is some parameter (wave velocity); $\mathrm{Re} = Vb/\nu$ is the Reynolds number. By requiring the functions $\tilde{\vec{v}}$ and $\tilde{p}$ to satisfy the Navier–Stokes equation, we arrive at a system of equations in u, v, and p:

$$\frac{1}{\mathrm{Re}}(\triangle u - p_x) - \left[\frac{3}{2}(1 - y^2) - c\right] u_x + 3yv = uu_x + vu_y; \tag{1.1}$$
$$\frac{1}{\mathrm{Re}}(\triangle v - p_y) - \left[\frac{3}{2}(1 - y^2) - c\right] v_x = uv_x + vv_y,\ u_x + v_y = 0.$$

We seek a solution to the system (1.1) in the domain $\Omega = \{x, y : |x| < \infty, \eta(x) < y < 1\}$. The line y = 1 corresponds to a rigid wall, on which is specified the no-slip condition

$$u = v = 0 \text{ for } y = 1. \tag{1.2}$$

The line $y = \eta(x)$ is a free boundary. On this line the following conditions are imposed:

$$\left\{\left[\frac{3}{2}(1 - \eta^2) - c + u\right]\eta' - v\right\}_{|y=\eta(x)} = 0; \tag{1.3}$$

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$$[(1-\eta'^2)(u_y+v_x-3\eta)+2\eta'(v_y-u_x)]_{y=\eta(x)}=0; \tag{1.4}$$

$$[-p+2(1+\eta'^2)^{-1}[v_y-\eta'(u_y+v_x-3\eta)+\eta'^2u_x]]_{y=\eta(x)}= \\ =-\mathrm{Re}W^{-1}(1+\eta'^2)^{-3/2}\eta''. \tag{1.5}$$

Here $W=\rho V^2b/\sigma$ is the Weber number (a dimensionless parameter, inversely proportional to the coefficient of surface tension σ), $\eta' = d\eta/dx$, and $\eta'' = d^2\eta/dx^2$. Condition (1.3) is a consequence of the kinematic conditions at the free boundary. Condition (1.4) signifies the absence of any tangential stress at the free boundary, and condition (1.5) is the equality of the normal stress and the capillary pressure.

In addition, we require periodicity in x of the solutions to the system (1.1) and the function η,

$$u(x+h, y)=u(x, y);\ v(x+h,y)=v(x, y);\ p(x+h,y)=p(x,y);\ \eta(x+h)=\eta(x), \tag{1.6}$$

and also the condition

$$\int_0^h \eta(x)\,dx=0, \tag{1.7}$$

which fixes the mean depth of the fluid. Condition (1.7) enables us to eliminate the trivial solutions to the problem (1.1)-(1.6), in which v = p = 0 and η = const. Besides those conditions listed, we should also include in the conditions of the problem the condition $\eta(x) < 1$, ensuring the absence of contact between the free surface and the bottom. However, we shall find below only small solutions of the traveling-wave type branching from the fundamental solution, and for these solutions the condition indicated will obviously be satisfied.

The mathematical problem consists of finding a function η and a solution u, v, p to Eqs. (1.1) in the domain Ω, such that the conditions (1.2)-(1.7) are satisfied. For any values of the parameters Re, W, c, and h, the problem (1.1)-(1.7) has the trivial solution $\eta = 0$, u = v = p = 0. The purpose of the present paper is to prove the existence of nontrivial solutions to this problem.

The nontrivial solutions to the problem (1.1)-(1.7) will be called rolling waves. Approximate theories of rolling waves, based on various approximate solutions of this problem, have been given by several authors [1-5]. In each of these treatments a one-parameter family (up to a shift in x) of approximate solutions to the problem (1.1)-(1.7) is constructed. In the present paper we establish the existence of a one-parameter family of solutions to the problem of rolling waves in the exact formulation.

2. AUXILIARY PROBLEM WITH FIXED BOUNDARY

For the investigation of the problem (1.1)-(1.7) we use a variation of the decomposition method set forth earlier [6]. We denote by $C_h^{l+\alpha}(\overline{\Omega})$ the subspace of functions $\varphi(x, y)$ of Hölder class $C^{l+\alpha}$ in the domain $\overline{\Omega}$ and periodic in x with period h ($l \geq 0$ is an integer, and $0 < \alpha < 1$). We denote by $C_{h,0}^{l+\alpha}(\mathrm{Re}^1)$ ($C_{h,0}^{l+\alpha}$ for short) the subspace of h-periodic functions with zero mean in the space $C^{l+\alpha}(\mathrm{Re}^1)$ (here Re^1 is the real axis). For fixed $\eta(x)$ we consider the problem of determining a solution $\vec{v}$, p to the system (1.1) in the domain Ω, satisfying the conditions (1.2)-(1.4), (1.6), and the additional condition

$$\int_0^h \int_{\eta(x)}^1 p_0\,dx\,dy=0 \tag{2.1}$$

[we call this the auxiliary problem with respect to the initial problem (1.1)-(1.7)].

<u>LEMMA 2.1.</u> There exists ε ($0 < \varepsilon < 1$) and $\mathrm{Re}_0 > 0$, such that for $\eta \in C_{h,0}^{3+\alpha}$, $|\eta|^{(3+\alpha)} \leqslant \varepsilon$ and $\mathrm{Re} \in [0, \mathrm{Re}_0]$ the problem (1.1)-(1.4), (1.6), (2.1) has a solution $\vec{v} \in \vec{C}_h^{2+\alpha}(\overline{\Omega})$, $p_0 \in C_h^{1+\alpha}(\overline{\Omega})$; this solution is unique in some ball $|\vec{v}|_{\Omega}^{(2+\alpha)} + |p_0|_{\Omega}^{(1+\alpha)} \leqslant \mathrm{const}$ in the space $\vec{C}_h^{2+\alpha}(\overline{\Omega}) \times C_h^{1+\alpha}(\overline{\Omega})$.

Hereafter the expressions $|\cdot|^{(l+\alpha)}$ and $|\cdot|_{\Omega}^{(l+\alpha)}$ denote the appropriate Hölder norms; the notation $\vec{v} \in \vec{C}_h^{l+\alpha}(\overline{\Omega})$ means that every component of the vector $\vec{v}$ belongs to the space $C_h^{l+\alpha}(\overline{\Omega})$.

We give the highlights of the proof of the lemma. Let us map the domain Ω onto the strip $\Pi = \{z_1, z_2 : |z_1| < \infty,\ 0 < z_2 < 1\}$ in the z_1z_2 plane by means of the transformation

$$z_1 = x, \quad z_2 = \frac{y - \eta(x)}{1 - \eta(x)}.$$

Because of (1.1) the functions $\vec{u}(z_1, z_2) = \vec{v}(x, y)$ and $q(z_1, z_2) = p_0(x, y)$ in the strip Π satisfy the system of equations

$$\Delta \vec{u} + \mathrm{Re}(\vec{a} \cdot \nabla \vec{u} + \vec{u} \cdot \nabla \vec{a}) - \nabla q = \vec{f}, \quad \nabla \cdot \vec{u} = f_3, \tag{2.2}$$

where $\vec{a} = (c - 3(1 - z_2^2)/2,\ 0)$; ∇ and Δ are the gradient and Laplacian with respect to the variables z_1 and z_2. The conditions (1.2)-(1.4), (1.6), and (2.1) generate the following boundary conditions for the system (2.2):

$$\vec{u} = 0 \quad \text{for} \quad z_2 = 1; \tag{2.3}$$

$$u_2 + (c - 3/2)\eta' = f_4 \quad \text{for} \quad z_2 = 0; \tag{2.4}$$

$$\frac{\partial u_1}{\partial z_2} + \frac{\partial u_2}{\partial z_1} - 3\eta = f_5 \quad \text{for} \quad z_2 = 0; \tag{2.5}$$

$$\vec{u}(z_1 + h, z_2) = \vec{u}(z_1, z_2), \quad q(z_1 + h, z_2) = q(z_1, z_2); \tag{2.6}$$

$$\int_0^h \int_0^1 q\, dz_1 dz_2 = \varkappa. \tag{2.7}$$

We do not give here the expressions for the components f_1, f_2 of the vector $\vec{f}$, the functions f_3, f_4, f_5, and the constant $\varkappa$. The important thing here is only that for $\vec{u} \in \vec{C}_h^{2+\alpha}(\overline{\Pi})$, $q \in C_h^{1+\alpha}(\overline{\Pi})$, $\eta \in C_{h,0}^{3+\alpha}(|\eta|^{(3+\alpha)} \leqslant \varepsilon_0 < 1)$, $\mathrm{Re} \in [0, \mathrm{Re}_0]$ and $c \in [-N, N](|N| < \infty)$ the relations $f_5 \in C_h^{1+\alpha}(\mathrm{Re}^1)$, $\vec{f} \in \vec{C}_h^{\alpha}(\overline{\Pi})$, $f_3 \in C_h^{1+\alpha}(\overline{\Pi})$, $f_4 \in C_h^{2+\alpha}(\mathrm{Re}^1)$ hold along with the estimates

$$|\vec{f}|_{\Pi}^{(\alpha)} \leqslant C_1 |\eta|^{(3+\alpha)} (|\vec{u}|_{\Pi}^{(2+\alpha)} + |q|_{\Pi}^{(1+\alpha)} + C_1(|\vec{u}|_{\Pi}^{(2+\alpha)})^2; \tag{2.8}$$
$$|f_3|_{\Pi}^{(1+\alpha)} + |f_4|_{\mathrm{Re}^1}^{(2+\alpha)} + |f_5|_{\mathrm{Re}^1}^{(1+\alpha)} + |\varkappa| \leqslant C_2 |\eta|^{(3+\alpha)} |\vec{u}|_{\Pi}^{(2+\alpha)} + C_2(|\eta|^{(3+\alpha)})^2,$$

where C_1 and C_2 depend only on ε_0, Re_0, and N (hereafter the symbols C_k, k = 1, 2, 3,... denote positive constants). In addition, if the function $\overline{f}_3$ corresponds to the solution $\vec{\overline{u}}, \vec{q}$ of the auxiliary problem, in which η is replaced by $\overline{\eta} \in C_{h,0}^{3+\alpha}$, $|\overline{\eta}|^{(3+\alpha)} \leqslant \varepsilon_0$, then

$$|f_3 - \overline{f}_3|_{\Pi}^{(1+\alpha)} \leqslant C_3 |\eta - \overline{\eta}|^{(3+\alpha)} (|\vec{u}|_{\Pi}^{(\alpha)} + |\vec{\overline{u}}|_{\Pi}^{(\alpha)} + C_3 |\vec{u} - \vec{\overline{u}}|_{\Pi}^{(\alpha)} (|\eta|^{(3+\alpha)} + |\overline{\eta}|^{(3+\alpha)}), \tag{2.9}$$

where C_3 depends only on ε_0, Re_0, and N. The functional $\varkappa$ and the differential expressions $\vec{f}$, f_4, and f_5, treated as operators on η, $\vec{u}$, q, have analogous Lipshitz continuity properties.

The solvability of the problem (2.2)-(2.7) for small η is proved by the method of successive approximations. For the starting approximation u^0, q^0 we take the solution to the linear problem

$$\nabla \vec{u}^0 + \mathrm{Re}(a \cdot \nabla \vec{u}^0 + \vec{u}^0 \cdot \nabla a) - \Delta q^0 = 0, \quad \nabla \cdot \vec{u}^0 = 0; \tag{2.10}$$

$$\vec{u}^0 = 0 \quad \text{for} \quad z_2 = 1; \tag{2.11}$$

$$u_2^0 + (c - 3/2)\eta' = 0 \quad \text{for} \quad z_2 = 0; \tag{2.12}$$

$$\frac{\partial u_1^0}{\partial z_2} + \frac{\partial u_2^0}{\partial z_1} - 3\eta = 0 \quad \text{for} \quad z_2 = 0; \tag{2.13}$$

$$\vec{u}^0(z_1+h, z_2)=\vec{u}^0(z_1,z_2), \quad q^0(z_1+h, z_2)=q^0(z_1, z_2); \tag{2.14}$$

$$\int_0^h \int_0^1 q^0 dz_1 dz_2 = 0. \tag{2.15}$$

Let $\vec{w}$ be an arbitrary solenoidal vector-valued function, h-periodic in z_1, of class $\vec{C}^2(\overline{\Pi})$, satisfying conditions (2.11) and $w_2 = 0$, $\partial w_1/\partial z_2 + \partial w_2/\partial z_1 = 0$ for $z_2 = 0$; q is an arbitrary function, h-periodic in z_1, of class $C^1(\overline{\Pi})$, and ω is the right triangle $0 < z_1 < h$, $0 < z_2 < 1$. We have the inequalities

$$-\int_\omega [\Delta\vec{w} + \mathrm{Re}(\vec{a}\cdot\nabla\vec{w} + \vec{w}\cdot\nabla\vec{a}) - \nabla q]\cdot\vec{w}dz \geqslant$$

$$\geqslant \int_\omega \left[2\sum_{j,k=1}^{2}\left(\frac{\partial w_j}{\partial z_k} + \frac{\partial w_k}{\partial z_j}\right)^2 - 3|\vec{w}|^2\right]dz \geqslant$$

$$\geqslant (2 - 3\mathrm{Re}C_4C_5)C_4(1 + C_5)^{-1}(\|\vec{w}\|_{2,\omega}^{(1)})^2,$$

where $C_4(h)$ and $C_5(h)$ are the constants in the Cornat and Poincare inequalities for the domain ω [6]; $\|\cdot\|_{2,\omega}^{(1)}$ denotes the norm in the Sobolev space $W_2^1(\omega)$. We choose $\mathrm{Re}_0 = 1/2C_4C_5$, which is then fixed. Then for $\mathrm{Re}\in(0,\mathrm{Re}_0]$ the solution to the problem (2.10)-(2.15) admits an a priori energy estimate. The existence of an a priori estimate for $\|\vec{u}^0\|_{2,\omega}^{(1)}$ makes it possible to prove the existence and uniqueness of the generalized solution to this problem (the outline of the proof is similar to that given in [6]). Following the methods of Solonnikov and Shchadilov [7], we can show that for any $\eta\in C_{h,0}^{3+\alpha}$ the generalized solution $\vec{u}^0$ to the problem (2.10)-(2.15) belongs to the class $\vec{C}_h^{2+\alpha}(\overline{\Pi})$ (correspondingly, $q^0\in C_h^{1+\alpha}(\overline{\Pi})$) and that the following estimate holds:

$$|\vec{u}^0|_\Pi^{(2+\alpha)} + |q^0|_\Pi^{(1+\alpha)} \leqslant C_6|\eta|^{(3+\alpha)}. \tag{2.16}$$

The subsequent approximations $\vec{u}^{n+1}$, $q^{n+1}(n \geq 0)$ to the solution of the problem (2.2)-(2.7) are determined from linear inhomogeneous problems of the type (2.10)-(2.15). The right-hand sides of the respective equations and of the boundary conditions are obtained by substituting the functions $\vec{u}^n$, q^n into the expressions for $\vec{f},\ldots,\varkappa$. Proceeding from the initial estimate (2.16), the inequalities (2.8) and (2.9), and the analogous inequalities for $|\vec{f}-\vec{\bar{f}}|_\Pi^{(\alpha)},\ldots,|\varkappa-\bar{\varkappa}|$, we can prove the convergence of the sequence $\{\vec{u}^n, q^n\}$ to the solution of problem (2.2)-(2.7), if $|\eta|^{(3+\alpha)} \leq \varepsilon$ and $\varepsilon\in[0,\varepsilon_0]$, where $\varepsilon_0 > 0$ is sufficiently small. The assertion of the solvability of the auxiliary problem can be strengthened. In the formulation of Lemma 2.1 any number less than the critical Reynolds number for Poiseuille flow in a plane channel can be chosen for Re_0.

On the basis of Lemma 2.1 there is defined an operator A for $\mathrm{Re}\in[0,\mathrm{Re}_0]$ which associates with the function $\eta\in C_{h,0}^{3+\alpha}$ for $|\eta|^{3+\alpha} \leq \varepsilon$ the expression

$$A(\eta)=\{-p_0 + 2(1+\eta'^2)^{-1}[v_y - \eta'(u_y+v_x-3\eta)+\eta'^2u_x]\}|_{y=\eta(x)}, \tag{2.17}$$

where $A[\eta(x)]\in C_h^{1+\alpha}(\mathrm{Re}^1)$. We note that the function p_0 differs by a constant from the p component of the solution to problem (1.1)-(1.7), so that $p = p_0 + C$. Putting Eq. (2.17) into the as yet unused condition at the free boundary (1.5) and eliminating the constant C, we arrive at the relation

$$B(\eta) \equiv A(\eta) - \bar{A} + \mathrm{Re}W^{-1}(1+\eta'^2)^{-3/2}\eta''=0, \tag{2.18}$$

where $\bar{A}$ denotes the mean value of the function $A[\eta(x)]$ on the interval (0, h). The operator B is defined in the ball $|\eta(x)|^{3+\alpha} \leq \varepsilon$ of the space $C_{h,0}^{3+\alpha}$ and goes to the space $C_{h,0}^{1+\alpha}$. The operator equation (2.18) is equivalent to the problem of rolling waves (1.1)-(1.7).

We define the functions $\vec{v}^0 = (u^0, v^0)$ and p_0^0 by the relations

$$\vec{v}^0(x, y) = \vec{u}^0(z_1, z_2), \quad p_0^0(x, y) = q^0,$$

where $\vec{u}^0$, q^0 is a solution to (2.10)-(2.15). It can be seen that $\vec{v}^0$, p_0^0 determine the solution of the problem (1.1)-(1.4), (1.6), and (2.1) linearized in the vicinity of $\eta = 0$, in which conditions (1.3) and (1.4) are placed on the unperturbed free boundary $\eta = 0$. We note that $v^0 \in C_h^{2+\alpha}(\bar{\Pi})$, $p_0^0 \in C_h^{1+\alpha}(\bar{\Pi})$, if $\eta \in C_{h,0}^{3+\alpha}$. In addition, as a consequence of (2.10)-(2.15) and (1.7) the mean value of the functions $v_y^0(x, 0)$ and $p_0^0(x, 0)$ on the interval (0, h) is equal to zero. Since the mapping $\eta \to (\vec{v}^0, p_0^0)$ is linear, it defines a linear operator

$$L(\eta) = (-p_0^0 + 2v_y^0)|_{y=0}, \tag{2.19}$$

going from $C_{h,0}^{3+\alpha}$ to $C_{h,0}^{1+\alpha}$.

<u>LEMMA 2.2.</u> We take the Frechet derivative of the operator B in the ball $|\eta|_{h,0}^{(3+\alpha)} \le \varepsilon$. The Frechet derivative at zero B_0' is given by

$$B_0'(\eta) = L(\eta) + \mathrm{ReW}^{-1}\eta''. \tag{2.20}$$

The definition of the Frechet derivative and the equality $B(0) = 0$ imply the representation $B(\eta) = B_0'(\eta) + F(\eta)$, where $|F(\eta)|^{(1+\alpha)} = 0\,(|\eta|^{(3+\alpha)})$ for $|\eta|^{(3+\alpha)} \to 0$. We need more precise information about the operator F, which is contained in the following assertion.

<u>LEMMA 2.3.</u> For any $\eta, \zeta \in C_{h,0}^{3+\alpha}$, such that $|\eta|^{(3+\alpha)}$, $|\zeta|^{(3+\alpha)} \le \varepsilon$ we have the estimate

$$|F(\eta) - F(\zeta)|^{(1+\alpha)} \leqslant C_7|\eta - \zeta|^{(3+\alpha)}(|\eta|^{(3+\alpha)} + |\zeta|^{(3+\alpha)}), \tag{2.21}$$

where C_7 does not depend on η and ζ.

The proofs of the Lemmas (2.2) and (2.3) are simple, but intricate, and they are not given here. As a consequence of (2.20) and the definition of $F(\eta)$ the operator equation (2.18) can be rewritten in the form

$$L(\eta) + \mathrm{ReW}^{-1}\eta'' + F(\eta) = 0. \tag{2.22}$$

3. PROPERTIES OF THE LINEAR OPERATOR L

The further course of our examination of the problem of rolling waves consists of the reduction of the problem to an operator equation of the form $\eta = \Phi(\eta)$, where Φ is an operator which is differentiable in the ball $|\eta|^{(3+\alpha)} \le \varepsilon$ in the space $C_{h,0}^{3+\alpha}$; $\Phi(0) = 0$ and the Frechet derivative Φ_0' is a completely continuous operator. In order to carry out the reduction, it is necessary to study the properties of the operator L.

It will prove convenient to extend the operator L to complex-valued functions and treat it as an operator from ${}^{\mathrm{com}}C_{h,0}^{3+\alpha}$ to ${}^{\mathrm{com}}C_{h,0}^{1+\alpha}$ [${}^{\mathrm{com}}C_{h,0}^{l+\alpha}$ is the space of functions of the form $\varphi(x) + i\psi(x)$, where φ and ψ are real functions of the class $C_{h,0}^{l+\alpha}$]. Furthermore, n denotes a nonzero integer and $\beta = 2\pi/h$.

<u>LEMMA 3.1.</u> The functions $\exp(in\beta x)$ are eigenfunctions of the operator L.

For the proof we note that if $\eta = \exp(in\beta x)$, then because of (2.10)-(2.15) the functions v^0 and p_0^0 are of the form

$$v^0 = \chi_n(y)\exp(in\beta x), \quad p_0^0 = \gamma_n(y)\exp(in\beta x).$$

The assertion of the lemma follows from here and the definition (2.19) of the operator L.

Thus, $L[\exp(in\beta x)] = \lambda_n \exp(in\beta x)$. Separating the variables in (2.10)-(2.15), we get a representation of the eigenvalues λ_n of L in the form

$$\lambda_n = -\frac{1}{(n\beta)^2}\dddot{\varphi}(0) + 3\dot{\varphi}(0) - \frac{i\mathrm{Res}}{n\beta}\dot{\varphi}(0), \tag{3.1}$$

where $\varphi = \varphi(y, \mathrm{Re}, s, n\beta)$ is the solution to the Orr–Sommerfeld equation

$$\ddot{\ddot{\varphi}} - 2k^2\ddot{\varphi} + k^4\varphi + ik\mathrm{Re}[(s + 3y^2/2)(\ddot{\varphi} - k^2\varphi) - 3\varphi] = 0, \tag{3.2}$$

satisfying the boundary conditions

$$\varphi(0)=-iks,\ \ddot{\varphi}(0)=-3ik+ik^3s,\ \varphi(1)=\dot{\varphi}(1)=0, \tag{3.3}$$

and we have introduced the notation $s = c-3/2$, $k = n\beta$; a dot denotes differentiation with respect to y.

We denote by L_0 the operator L with Re = 0 and we set $\delta L = L - L_0$. The eigenvalues λ_{n0} of L_0 can be calculated explicitly:

$$\lambda_{n0}=\frac{ik^2}{\operatorname{ch} k \operatorname{sh} k-k}[2s(\operatorname{ch}^2 k+k^2)-3] \tag{3.4}$$

($k = n\beta$). In view of the reality of s, all λ_{n0} are purely imaginary. For $n \to \infty$ we have the representation

$$\lambda_{n0}=2is(n\beta)^2 \operatorname{sign} n+O(e^{-|n|\beta}). \tag{3.5}$$

The operator L differs from L_0 by subordinate terms, which are small along with Re. It can be proved that for $\mathrm{Re}\in[0,\mathrm{Re}_0]$ and for any integer $n \neq 0$ the estimate

$$|\lambda_n-\lambda_{n0}|\leqslant C_8 \mathrm{Re}(|n|\beta+1), \tag{3.6}$$

holds, where C_8 does not depend on Re and s for $0 \le \mathrm{Re} \le \mathrm{Re}_0$ and $|s| \le N$ (N is any positive number). The proof of the estimate (3.6) is omitted. It uses standard techniques in the asymptotic expansion of solutions to ordinary linear differential equations containing a large parameter (see, e.g., Wazow [8]).

LEMMA 3.2. There exists $\gamma > 0$ such that for $\mathrm{Re}\in[0,\mathrm{Re}_0]$, $|s| \le N$ the operator $B_0'-\gamma$ has an inverse $(B_0'-\gamma)^{-1}: C_{h,0}^{1+\alpha} \to C_{h,0}^{3+\alpha}$.

Proof. According to definition (2.20), $B_0' = L_0 + \mathrm{Re}W^{-1}d^2/dx^2 + \delta L$. The functions $\exp(in\beta x)$ are eigenfunctions of the operator $B_0'-\delta L$, to which correspond the eigenvalues $\lambda_{n,0}-\mathrm{Re}W^{-1}(n\beta)^2$. This together with (3.5) implies that if $(B_0'-\delta L)(f)\in {}^{\mathrm{com}}C_{h,0}^{1+\alpha}$ and $f\in {}^{\mathrm{com}}C_{h,0}^{1+\alpha}$, then $f\in {}^{\mathrm{com}}C_{h,0}^{3+\alpha}$. From the definition of the operators L_0 and δL and the estimates of the solution to (2.10)-(2.15) in the Hölder class it follows that $\delta L(f)\in {}^{\mathrm{com}}C_{h,0}^{2+\alpha}$, if $f\in {}^{\mathrm{com}}C_{h,0}^{3+\alpha}$. This means that the operator $D = (B_0'-\delta L)^{-1}(\delta L-\gamma)$ is completely continuous in ${}^{\mathrm{com}}C_{h,0}^{3+\alpha}$, and the operator I + D is Fredholm. For I + D to be invertible, it suffices that there be no nontrivial solutions to the equation $(I + D)(f) = 0$. Every solution to this equation of class ${}^{\mathrm{com}}C_{h,0}^{3+\alpha}$ is representable in the form $\Sigma f_n \exp(in\beta x)$, where the coefficients f_n are subject to the conditions

$$[\lambda_n-\mathrm{Re}W^{-1}(n\beta)^2-\gamma]f_n=0.$$

Choosing $\gamma = C_8^2\mathrm{Re}_0W/2$ and taking into account (3.4) and (3.6), we find that all $f_n = 0$, if $\mathrm{Re}\in[0,\mathrm{Re}_0]$. In view of the identity $(B_0'-\delta L)(I + D) = B_0'-\gamma$, the existence of the operators $(I + D)^{-1}$ and $(B_0'-\delta L)^{-1}$ implies the existence of the operator $(B_0'-\gamma)^{-1}$: ${}^{\mathrm{com}}C_{h,0}^{1+\alpha} \to {}^{\mathrm{com}}C_{h,0}^{3+\alpha}$. To complete the proof of the lemma it remains to note that the operator $(B_0'-\gamma)^{-1}$ takes real functions into real functions.

It is clear that $(B_0'-\gamma)^{-1}$ can also be viewed as an operator acting in the space ${}^{\mathrm{com}}C_{h,0}^{3+\alpha}$. Proceeding from estimates of the Schauder type for the solutions to (2.10)-(2.15), we can establish the complete continuity of this operator in ${}^{\mathrm{com}}C_{h,0}^{3+\alpha}$ (and, consequently, in the real space $C_{h,0}^{3+\alpha}$). We denote by ${}^{\mathrm{com}}L_2'$ the subspace of the complex space ${}^{\mathrm{com}}L_2(0,h)$ consisting of the functions with zero mean value on the interval (0, h). The operator $(B_0'-\gamma)^{-1}$ can be extended to a completely continuous operator in ${}^{\mathrm{com}}L_2'$, because it has a complete system of eigenfunctions $\{\exp(in\beta x)\}$ in ${}^{\mathrm{com}}L_2'$, to which correspond the eigenvalues

$$\mu_n=[\lambda_n-\mathrm{Re}W^{-1}(n\beta)^2-\gamma]^{-1}. \tag{3.7}$$

The eigenvalues μ_n are complex. However, if μ_n is a real eigenvalue, then it is at least of multiplicity two. The corresponding (real) eigenfunctions are $\cos(n\beta x)$ and $\sin(n\beta x)$. The twofold degeneracy of the real spectrum of $(B_0'-\gamma)^{-1}$ is connected with the invariance of the problem (2.10)-(2.15) with respect to translation of x.

LEMMA 3.3. For sufficiently small Re > 0 there exist s and β such that $(B_0'-\gamma)^{-1}$ has a twofold eigenvalue $\mu_1 = -\gamma^{-1}$.

Proof. As a consequence of (3.7) the eigenvalue $\mu_1 = -\gamma^{-1}$ of the operator $(B_0'-\gamma)^{-1}$ corresponds to the eigenvalue $\lambda_1 = \mathrm{Re}W^{-1}\beta^2$ of the operator L. Let us show that for small Re the system equations in s and β

$$\begin{gathered}\mathrm{Real}\lambda_1(\mathrm{Re}, s, \beta) - \mathrm{Re}W^{-1}\beta^2 = 0;\\ \mathrm{Im}\lambda_1(\mathrm{Re}, s, \beta) = 0\end{gathered} \tag{3.8}$$

has a real solution.

Let us denote $\delta\lambda_1 = \lambda_1 - \lambda_{10}$, where λ_{10} is determined by Eq. (3.4), in which we have set $k = \beta$. From the definition (3.1)-(3.3) of the eigenvalues of the operator L comes the estimate

$$|\delta\lambda_1| \leqslant C_9\mathrm{Re},$$

where C_9 does not depend on Re, s, β for $0 \le \mathrm{Re} \le \mathrm{Re}_0$, $|s| \le N$, $0 \le \beta \le \beta_0$ (β_0 is an arbitrary positive number). In addition, from the definition of λ_n follows the representation $\lambda_n = \Lambda(\mathrm{Re}, s, n\beta)$, where Λ is some standard (smooth) function. From this and the estimate (3.6) we get the inequality

$$|\delta\lambda_1| \leqslant C_{10}\mathrm{Re}\beta \tag{3.9}$$

with $\mathrm{Re} \in [0, \mathrm{Re}_0]$, $|s| \leqslant N$, $\beta \geqslant \beta_0$, where C_{10} does not depend on Re, s, and β.

For fixed $\beta \ge 0$ let us consider the second of the relations (3.8) as an equation with respect to s. This equation can be written in the form

$$s = \frac{3}{2(\mathrm{ch}^2\beta + \beta^2)} - \frac{\mathrm{ch}\,\beta\,\mathrm{sh}\,\beta - \beta}{2\beta^2(\mathrm{ch}^2\beta + \beta^2)}\mathrm{Im}\delta\lambda_1(\mathrm{Re}, s, \beta). \tag{3.10}$$

Taking (3.6) into account, we get an a priori estimate of the solutions to (3.10):

$$\left|s - \frac{3}{2(\mathrm{ch}^2\beta + \beta^2)}\right| \leqslant \frac{C_{11}C_8(N)\,\mathrm{Re}\,\beta}{\beta^2+1}, \tag{3.11}$$

where C_{11} is some absolute constant. Let us choose some fixed $N \ge 2$. Then for $0 \le \mathrm{Re} \le \mathrm{Re}_1 = \min(\mathrm{Re}_0, 1/C_{11}C_8(N))$ and any $\beta \ge 0$ we have $|s| \le 2$.

In order to prove the existence of a solution to (3.10), we have to estimate the magnitude of the derivative $\partial(\mathrm{Im}\delta\lambda_1)/\partial s$. It turns out that for any $\beta \geqslant 0$, $\mathrm{Re} \in [0, \mathrm{Re}_0]$ and s, $|s| \le N$, the following estimate is valid:

$$\left|\frac{\partial}{\partial s}(Im\delta\lambda_1)\right| \leqslant C_{12}(N)\,\mathrm{Re}. \tag{3.12}$$

The proof of this estimate is not complicated but cumbersome and is not given here. In view of (3.11) and (3.12) there exists Re_2 $(0 < \mathrm{Re}_2 < \mathrm{Re}_1)$ such that for $\mathrm{Re} \in [0, \mathrm{Re}_2]$ and any $\beta \ge 0$ Eq. (3.10) has a unique solution s_* on the interval $|s| \le 2$. This solution is a continuous function of the parameters Re and β.

We introduce the notation

$$\xi(\mathrm{Re}, \beta) = \mathrm{Real}\lambda_1[\mathrm{Re}, s_*(\mathrm{Re}, \beta), \beta] - \mathrm{Re}W^{-1}\beta^2, \tag{3.13}$$

and rewrite the first equation in (3.8) in the form

$$\xi(\mathrm{Re}, \beta) = 0. \tag{3.14}$$

The function ξ is continuous in the domain $0 \le \mathrm{Re} \le \mathrm{Re}_2$, $\beta \ge 0$, and $\xi(0, \beta) = 0$. Equations (3.13), (3.4), and (3.9) indicate that $\xi \to -\infty$ for $\beta \to -\infty$ and any $W > 0$, $\mathrm{Re} \in (0, \mathrm{Re}_2)$. Thus, for the solvability of Eq. (3.14) for small Re it is sufficient to establish that $\xi(\mathrm{Re}, 0) > 0$.

For the proof of this fact we employ the representation implied by Eqs. (3.1)-(3.3):

$$\text{Real}\lambda_1 = \text{Re}\lambda_{11} + \tau(\text{Re}, \beta), \tag{3.15}$$

where $\tau = O(\text{Re}^2)$ for $\text{Re} \to 0, \beta \in [0, \beta_0]$ and $\lambda_{11}(\beta) = -(1/\beta^2)\dddot{\psi}(0) + 3\dot{\psi}(0) - (is_0/\beta)\,\dot{\psi}_0(0)$. Here the function $\varphi_0(y)$ is a solution to the boundary-value problem (3.1), (3.2) with Re = 0, $s_0 = s_*(0, \beta) = 3/2\,(\text{ch}^2\beta + \beta^2)$ (3.10), (3.11), while $\psi(y)$ satisfies the equation

$$\ddddot{\psi} - 2\beta^2\ddot{\psi} + \beta^4\psi = -i\beta[(s_0 + 3y^2/2)\ (\ddot{\varphi}_0 - \beta^2\varphi_0) - 3\varphi_0]$$

and the boundary conditions $\psi(0) = \ddot{\psi}(0) = \psi(1) = \dot{\psi}(1) = 0$ (note that the function φ_0 assumes purely imaginary values). Simple computations give the value $\lambda_{11}(0) = 18/5$. By virtue of (3.15), (3.13), and the estimate $\tau = O(\text{Re}^2)$ we conclude that $\xi(\text{Re}, 0) > 0$ for $0 < \text{Re} \leq \text{Re}_3$, where $\text{Re}_3 \leq \text{Re}_2$ is sufficiently small. This proves the solvability of Eq. (3.14) and, hence, of the system (3.8).

Let $\text{Re} \in (0, \text{Re}_3]$ and s_*, β_* be a solution to the system (3.8). We shall show that for small Re the identity $\lambda_n \equiv \Lambda(\text{Re}, s_*, n\beta_*) = \text{ReW}^{-1}\beta_*^2$ cannot be satisfied for any n not equal to 1 or −1. As a consequence of Eqs. (3.4), (3.6), and (3.11) we have

$$\text{Im}\,\lambda_n = \frac{3\,(n\beta_*)^2}{\text{ch}\,(n\beta_*)\,\text{sh}\,(n\beta_*) - n\beta_*}\left[\frac{\text{ch}^2\,(n\beta_*) + (n\beta_*)^2}{\text{ch}^2\,\beta_* + \beta_*^2} - 1\right] + \theta\,(\text{Re}, n),$$

where the function θ is estimated by $|\theta| \leq C_{13}\text{Re}\,[n]^2$ with some constant C_{13}. If $|n| > 1$ and $\text{Re} \leq \text{Re}_3$ is sufficiently small, then $\text{Im}\,\lambda_n \neq 0$, which implies the desired result. From the inequality $\lambda_n \neq \text{ReW}^{-1}\beta_*^2$ in view of (3.7) it follows that $\mu_n \neq -\gamma^{-1}$ for $[n] \neq 1$. This means that the number $-\gamma^{-1}$ is no more than a twofold eigenvalue of the operator $(B_0' - \gamma)^{-1}$. Since it is real, it has precisely multiplicity two. Lemma 3.3 is proved.

We note that the system (3.8) was considered by Yih [9], who analyzed the stability in the linear approximation of the flow down an inclined plane. However, in that paper the parameter β was assumed given, while the unknown quantity s was not assumed real. Goncharenko and Urintsev [10] investigated the stability of this problem in a broad range of the decisive parameters.

4. EXISTENCE OF ROLLING WAVES

THEOREM 4.1. There exists $\text{Re}_* > 0$ such that for $0 < \text{Re} \leq \text{Re}_*$ the problem (1.1)-(1.7) has a one-parameter (up to a shift in x) family of solutions.

Proof. Choose Re_* such that for $\text{Re} \in (0, \text{Re}_*]$ the assertion of Lemma 3.3 is satisfied.

As shown in Sec. 2 the problem of the bifurcation of (1.1)-(1.7) is equivalent to the operator equation (2.22). This equation can be brought to the form

$$\eta = -\gamma\,(B_0' - \gamma)^{-1}(\eta) - (B_0' - \gamma)^{-1}F(\eta). \tag{4.1}$$

We shall prove the existence of nontrivial solutions to Eq. (4.1). Note that if $\eta(x)$ is a solution to this equation, then $\eta(x + a)$ will also be a solution for any a = const. This follows from the invariance of Eqs. (1.1) and conditions (1.2)-(1.7) under translation with respect to x.

The operators $(B_0' - \gamma)^{-1}$ and F depend continuously on the parameters Re, W, s = c − 3/2, and $\beta = 2\pi/h$. We fix $\text{Re} \in (0, \text{Re}_*]$, and $W_0 > 0$ and choose for s and β the solution to Eqs. (3.8) corresponding to the given Re and W_0. The parameter W remains at our disposal. We denote

$$Q = -\gamma\,(B_0' - \gamma)^{-1}_{|W=W_0};\quad \delta Q = -\gamma\,(B_0' - \gamma) - Q;\quad T = -(B_0' - \gamma)^{-1}F,$$

and write Eq. (4.1) in the form

$$\eta = Q(\eta) + \delta Q(\eta) + T(\eta). \tag{4.2}$$

From the definition of the nonlinear operator T, the inequality (2.21), and the claim of Lemma 3.2 there follows the estimate

$$|T(\eta) - T(\zeta)|^{(3+\alpha)} \leqslant C_{14}|\eta-\zeta|^{(3+\alpha)}(|\eta|^{(3+\alpha)}+|\zeta|^{(3+\alpha)}) \tag{4.3}$$

for any $\eta, \zeta \in C_{h,0}^{3+\alpha} \text{ c } |\eta|^{(3+\alpha)}, \ |\zeta|^{(3+\alpha)} \leqslant \varepsilon$ and any $W \in [W_1, W_2], 0 \leq W_1 < W_0 < W_2 < \infty$($C_{14}$ is independent of η, ζ, and W). The linear operator Q and δQ are completely continuous in $C_{h,0}^{3+\alpha}$ (see Sec. 3). The definition of δQ and the properties of the operator B_0' imply the inequality

$$|\delta Q(\eta)|^{(3+\alpha)} \leqslant C_{15}|W-W_0||\eta|^{(3+\alpha)} \tag{4.4}$$

for any $\eta \in C_{h,0}^{3+\alpha}$, $W \in [W_1, W_2]$.

The linear operator Q has the twofold eigenvalue 1 with corresponding eigenfunctions $\cos\beta x$ and $\sin\beta x$. It can be seen that these functions are also eigenfunctions of the adjoint operator. We introduce into consideration the operator S acting according to the rule

$$S(\eta)=Q(\eta) - \pi^{-1}\beta[(\eta, \cos\beta x)\cos\beta x+(\eta, \sin\beta x)\sin\beta x]$$

[the symbol (,) denotes the scalar product in $L_2(0, h)$]. According to the generalized Schmidt lemma (see, e.g., Vainberg and Trenogin [11]) unity is not an eigenvalue of S. Setting

$$\pi^{-1}\beta(\eta, \cos\beta x)=\xi_1, \ \pi^{-1}\beta(\eta, \sin\beta x)=\xi_2, \tag{4.5}$$

we obtain from (4.2) the equation

$$(I - S)(\eta)=\xi_1\cos\beta x+\xi_2\sin\beta x+\delta Q(\eta)+T(\eta). \tag{4.6}$$

From the boundedness of $(I-S)^{-1}$ and the estimates (4.3) and (4.4) we conclude that for sufficiently small $|\vec{\xi}| = (\xi_1^2 + \xi_2^2)^{1/2}$ and $|W-W_0|$ Eq. (4.6) has a unique solution $\eta \in C_{h,0}^{3+\alpha}$, such that $|\eta|^{3+\alpha} \to 0$ when $\vec{\xi} \to 0$ and $W \to W_0$.

To determine the possible values of $\vec{\xi} = (\xi_1, \xi_2)$ it is necessary to substitute the obtained expression for $\eta(x)$ into Eqs. (4.5), which leads to the system of bifurcation equations

$$\varphi_1(\xi_1, \xi_2, W)=\xi_1, \ \varphi_2(\xi_1, \xi_2, W)=\xi_2. \tag{4.7}$$

It turns out that the system (4.7) can be reduced to a single equation relating only ξ_1 and W. For this we use the invariance of Eq. (4.2) with respect to translations in x. This invariance implies the group property of Eq. (4.6), represented by the transformation

$$x \to x+a, \ \xi_1 \to \xi_1\cos\beta a - \xi_2\sin\beta a,$$
$$\xi_2 \to \xi_1\sin\beta a+\xi_2\cos\beta a, \ \eta \to \eta.$$

This means that if Eqs. (4.7) have a nontrivial solution ξ_1, ξ_2, then they have a one-parameter family of solutions obtained from the given solution by a rotation of the vector $\vec{\xi}$ through an arbitrary angle. Therefore, we can set $\xi_2 = 0$ at the start and consider instead of (4.7) the equation

$$\varphi_1(\xi_1, 0, W)=\xi_1. \tag{4.8}$$

For fixed $|\vec{\xi}|$ we thereby single out one of the solutions to Eq. (4.6), for which $(\eta, \sin\beta x) = 0$. The possibility of reducing the system of bifurcation equations by using the group properties of the branching problem was established by Loginov and Trenogin [12].

In Eq. (4.8) it is convenient to consider ξ_1 as given and W as unknown. A similar method was used by Yudovich [13]. The definition of φ_1 implies the representation $\varphi_1(\xi_1, 0, W) = \xi_1[1 + r(\xi_1, W)]$, where r is a smooth function, while $r(0, W_0) = 0$. For local solution of Eq. (4.8) with respect to W, it is sufficient to verify that $r_W(0, W_0) \neq 0$. Computation yields

$$r_W(0, W_0) = -\gamma \frac{\partial \mu_1}{\partial W}\Big|_{W=W_0} = \frac{\mathrm{Re}\,\beta^2}{\gamma W_0^2} > 0$$

[here μ_1 is the first eigenvalue of $(B_0' - \gamma)^{-1}$, determined by Eq. (3.7), and the positive parameters Re and β were previously fixed]. Solvability of Eq. (4.8) means that for any W sufficiently close to W_0, Eq. (4.2) has a nontrivial solution $\eta \in C^{3+\alpha}_{h,0}$, with $|\eta|^{3+\alpha} \to 0$ for $W \to W_0$. In view of the arbitrariness of $W_0 > 0$ it follows that for any fixed $\mathrm{Re} \in (0, \mathrm{Re}_*]$ and $W > 0$ up to a shift in x Eq. (4.1) has a one-parameter family of small solutions ($|\eta|^{3+\alpha}$ may be taken as the parameter). We have thus established the existence of a one-parameter family of solutions to the problem (1.1)-(1.7).

In conclusion, we note that the bifurcation problem considered here is a very particular case of the general problem of rolling waves. Of undisputed interest would be the generalization of Theorem (4.1) to the case of arbitrary Reynolds number and also the proof of the existence of rolling in an inclined plane. The main obstacle in the way is the lack at present of any analog to Lemma (3.3) in these cases.

Even more difficult is the problem of the existence of three-dimensional spatially periodic motions in a fluid layer flowing down an inclined plane. We note that in the case of a vertical plane this problem has been positively solved by Nepomnyashchii [14] in the long-wavelength approximation.

In the present paper we have not touched upon the question of stability of rolling waves. The stability of the wave regimes of flow down an inclined plane was studied in the linear approximation by Shkadov [15] and Nepomnyashchii [16, 17]. The stability of the wave motions of a viscous fluid in the exact formulation is a completely open question.

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